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Path integral methods via the use of the central limit theorem and application

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Abstract
We consider a path integral in the phase space possibly with an influence functional in it and we use a method based on the use of the central limit theorem on the phase of the path integral representation to extract an equivalent expression which can be used in numerical calculations. Moreover we give conditions under which we can extract closed analytical results. As a specific application we consider a general system of two coupled and forced harmonic oscillators with coupling of the form $x_1 x_2^{\alpha}$ and we derive the relevant sign solved propagator.

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1. Introduction

Path integral methods appear in various areas of physics including quantum mechanics, quantum field theory and polymer physics. They constitute a powerful and elegant method of treatment although their mathematical foundation constitutes a large, complex and difficult topic [1].

In the present paper, we intend to give methods based on the use of the central limit theorem [2, 3] in the phase of the path integral expressions to handle path integral expressions in real time and derive expressions free from the sign problem [4], a central problem appearing in the real-time Monte Carlo integration [5]. The sign problem is due to a phase or sign appearing in the quantum Monte Carlo expression, which changes in a periodic way. Therefore the terms of different sign cancel each other and the Monte Carlo error increases. The present result appears free of the sign problem.

Further we give conditions to extract exact closed analytical expressions for the propagators. The evaluated propagators are called 'sign solved propagators' (SSP) due to the origin [4] of the present method. We note that they satisfy the semigroup property but they are not directly related to the zeta function due to the presence of a delta function (see the theorem at the end of section 2). In fact knowledge of the zeta function requires knowledge of the eigenvalues of the whole system, something that cannot be extracted via the present formalism.

Previously we have considered [6] a model involving an atom in an over-damped cavity and after deriving its influence functional we have studied its internal dynamics by applying to that influence functional methods similar to those developed in the present paper. In another publication [7] we developed a path integral model for the study of the interaction of atoms with ultra short laser pulses and derived expressions from which the exact SSP can be extracted as the number of the time slices tends to infinity.

We note that coherent spaces constitute another area of application of the combined use of path integral methods with the central limit theorem [8]. We have not used the naming sign solved propagator in the main body of [8].

Here as an application of the present theory we consider the case of two coupled and driven harmonic oscillators of time-dependent frequencies as well as masses and coupling of the form $x_1x_2^{\alpha}$ and derive the corresponding SSP.

The present paper proceeds as follows. In section 2, we present the theory and the major results of the application of the central limit theorem in the phase of the path integral expression. In section 3, we present an application of two time-dependent coupled and driven harmonic oscillators. We derive the corresponding SSP and finally we make more specific calculations of the SSP in the case of two coupled and forced oscillators with damping. Finally, in section 4 we present our conclusions.

2. Central limit theorem and path integral expressions

The central aim of the present section is to prove the theorem appearing at the end of the present section. Before proceeding with the computation of path integrals of interacting systems it is instructive to consider the path integral

$$K(r_f, r_i; t) = \int\!\!\int Dr \frac{Dp}{2\pi} \exp\left[\mathrm{i}\int_0^t \mathrm{d}\tau (p(\tau)\dot{r}(\tau) - H(p, r, \tau))\right]. \tag{1}$$

After the theorem of equations (15) we will introduce an influence functional in it. To proceed we consider its discrete expression of $N + 1$ time slices

$$K^{(N)}(r_f, r_i; t) = \prod_{n=1}^{N}\left[\int \mathrm{d}r_n\right]\prod_{n=1}^{N+1}\left[\int \frac{\mathrm{d}p_n}{2\pi}\right]\exp\left\{\mathrm{i}\sum_{n=1}^{N+1}[p_n(r_n - r_{n-1}) - \varepsilon H(p_n, r_n, t_n)]\right\}, \tag{2}$$

where

$$\varepsilon = \frac{t}{N+1}, \tag{3a}$$

$$t_n = n\varepsilon. \tag{3b}$$

We set $r_0 = r_i$ and $r_{N+1} = r_f$. In the present development, we generally consider the Hamiltonian H to have the form

$$H(p, r, t) = \frac{p^2}{2M} + V(r, t). \tag{4}$$

Now we proceed by considering from the theoretical point of view the application of the Monte Carlo theorem to the expression (2). At first we note that from the probabilistic point of view the configuration space variables are statistically independent, arbitrary and moreover equivalent if we use the same sampling distribution of the random variable X_r for all of them. The same is valid for the variables of the momentum group supposing the same sampling distribution of the random variable X_p for all of them as well. Further we assume that the random variables X_r and X_p are statistically independent. Consequently, their mutual covariance matrices are going to be zero and the covariance matrix of the multidimensional phase-space sampling distribution is going to have elements $\sigma_{ij}^{rr} = \sigma_r^2 \delta_{ij}$, $\sigma_{kl}^{pp} = \sigma_p^2 \delta_{kl}$ and $\sigma_{ik}^{rp} = \sigma_{ki}^{pr} = 0$ where the lower indices i, j correspond to the configuration space variables while the lower indices k, l correspond to the momentum space variables. Moreover the upper index r corresponds to the random variable X_r and the upper index p to the random variable X_p. σ_r^2 is the variance corresponding to each of the one-dimensional equivalent configuration space sampling distributions and σ_p^2 is the variance corresponding to each of the one-dimensional equivalent momentum space sampling distributions. δ_{ij} is the Kronecker's symbol.

Now according to the central limit theorem [2] the following convergence in distribution applies:

$$\sqrt{N+1}\left[\left(\frac{1}{N+1}\sum_n \begin{bmatrix} t\frac{p_n^2}{2M} \\ tV(r_n, t_n) \end{bmatrix}\right) - \begin{bmatrix} t\left\langle \frac{p^2}{2M}\right\rangle \\ t\langle V(t)\rangle \end{bmatrix}\right] \xrightarrow{D} Z. \tag{5}$$

For the time-dependent potential the mean value expression is given as

$$\langle V(t)\rangle = \frac{1}{N+1}\sum_{n=1}^{N+1} \langle V(r_n, t_n)\rangle. \tag{6}$$

In each term on the right-hand side of equation (6) the expectation value is taken with respect to the configuration space sampling distribution.

The continuous Gaussian random variable Z is defined on a space composed of $N+1$ copies of the phase space. Each vector on the left of equation (5) depends diagonally on the variables of just one of those copies. Z obeys a normal probability density with mean zero and diagonal covariance matrix

$$\overleftrightarrow{\Sigma} = \overleftrightarrow{\Sigma}_2 \otimes I_{N+1}, \tag{7a}$$

where

$$\overleftrightarrow{\Sigma}_2 = \begin{pmatrix} t^2\sigma_m^2 & 0 \\ 0 & t^2\sigma_V^2(t) \end{pmatrix}. \tag{7b}$$

I_{N+1} is the $N+1$-dimensional unit matrix and moreover on the one hand

$$\sigma_m^2 = \mathrm{Cov}\left(\frac{p^2}{2M}, \frac{p^2}{2M}\right), \tag{7c}$$

where the covariance is calculated with respect to the momentum space sampling distribution, and on the other upon taking into account definition (6) we have set

$$\sigma_V^2(t) = \frac{\sum_{n=1}^{N+1} \langle (V(r_n, t_n) - \langle V(t)\rangle)^2\rangle}{N+1}, \tag{7d}$$

where the expectation values are calculated with respect to the configuration space sampling distribution.

We note that in order to write the covariance matrix (7b) we have taken into account the fact that the random variables X_r and X_p and their corresponding functions are statistically independent.

Therefore the distribution density of the random variable Z in equation (5) is given by the expression

$$f_Z(\vec{x}) = \frac{1}{(2\pi)^{N+1}\sqrt{\det(\overleftrightarrow{\Sigma})}} \exp\left[-\vec{x}\frac{1}{2\overleftrightarrow{\Sigma}}\vec{x}^T\right], \tag{8}$$

where $\vec{x} = (p_1, p_2, \ldots, p_N, p_{N+1}, r_1, r_2, \ldots, r_N, r_{N+1})$.

We note that in equation (5) we have a sum of functions of independently distributed random vectors. Then for integrable functions with respect to the time their domain of attraction [9], according to an application of Lyapunov's theorem, is Gaussian and the above convergence in distribution applies.

Further according to a corollary of the Cramer's theorem [2] if we consider a function $F(x, y)$ then the following convergence in distribution applies:

$$\sqrt{N+1}\left[F\left(\frac{t}{N+1}\sum_n \frac{p_n^2}{2M}, \frac{t}{N+1}\sum_n V(r_n, t_n)\right) - F\left(t\left\langle\frac{p^2}{2M}\right\rangle, t\langle V(t)\rangle\right)\right] \xrightarrow{D} X, \tag{9}$$

where the continuous Gaussian random variable X is defined on a space composed of $N+1$ copies of the phase space. Each vector on the left of equation (9) depends diagonally on the variables of just one of those copies. X obeys a normal probability density with mean zero and diagonal covariance matrix

$$\left[F'_x\left(t\left\langle\frac{p^2}{2M}\right\rangle, t\langle V(t)\rangle\right)F'_y\left(t\left\langle\frac{p^2}{2M}\right\rangle, t\langle V(t)\rangle\right)\right]\overleftrightarrow{\Sigma}_2\begin{bmatrix}F'_x\left(t\left\langle\frac{p^2}{2M}\right\rangle, t\langle V(t)\rangle\right)\\ F'_y\left(t\left\langle\frac{p^2}{2M}\right\rangle, t\langle V(t)\rangle\right)\end{bmatrix}\otimes I_{N+1}. \tag{10}$$

Its corresponding distribution is given by a similar expression as in equation (8) but with the covariance matrix (10). Note that another way of looking at the above distributions is as products of one-dimensional distributions.

On considering the functions $F(x, y) = \cos(x + y)$, $F(x, y) = \sin(x + y)$ and upon applying the above corollary of Cramer's theorem we obtain the following convergence in distribution:

$$\sqrt{N+1}\left[\exp\left(-\mathrm{i}\frac{t}{N+1}\sum_n\left(\frac{p_n^2}{2M} + V(r_n, t_n)\right)\right) - \exp\left(-\mathrm{i}t\langle H(t)\rangle\right)\right] \xrightarrow{D} Y - \mathrm{i}W, \tag{11}$$

where

$$\langle H(t)\rangle = \left\langle\frac{p^2}{2M}\right\rangle + \langle V(t)\rangle. \tag{12a}$$

The continuous Gaussian Y, W random variables are defined on a space composed of $N+1$ copies of the phase space. Each vector on the left-hand side of equation (11) depends diagonally on the variables of just one of those copies. Y, W obey normal probability densities with mean zero and respectively diagonal covariance matrices

$$[\sin(t\langle H(t)\rangle)\quad \sin(t\langle H(t)\rangle)]\overleftrightarrow{\Sigma}_2\begin{bmatrix}\sin(t\langle H(t)\rangle)\\ \sin(t\langle H(t)\rangle)\end{bmatrix}\otimes I_{N+1} \tag{12b}$$

and

$$[\cos(t\langle H(t)\rangle)\quad \cos(t\langle H(t)\rangle)]\overleftrightarrow{\Sigma}_2\begin{bmatrix}\cos(t\langle H(t)\rangle)\\ \cos(t\langle H(t)\rangle)\end{bmatrix}\otimes I_{N+1}. \tag{12c}$$

Now we can write the discrete expression

$$
\begin{aligned}
K^{(N)}(r_f, r_i; t) = {} & \frac{(\rho_2-\rho_1)^N}{M_1^N M_2^N} \frac{(q_2-q_1)^{N+1}}{(2\pi L_1 L_2)^{N+1}} \\
& \times \sum_{i_{11}=1}^{M_1} \cdots \sum_{i_{1n}=1}^{M_1} \cdots \sum_{i_{1N}=1}^{M_1} \sum_{j_{11}=1}^{L_1} \cdots \sum_{j_{1n}=1}^{L_1} \cdots \sum_{j_{1N+1}=1}^{L_1} \exp\left\{ \mathrm{i} \sum_{n=1}^{N+1} [p_{j_{1n}}(r_{i_{1n}} - r_{i_{1n-1}})] \right\} \\
& \times \Bigg(\sum_{i_{21}=1}^{M_2} \cdots \sum_{i_{2n}=1}^{M_2} \cdots \sum_{i_{2N}=1}^{M_2} \sum_{j_{21}=1}^{L_2} \cdots \sum_{j_{2n}=1}^{L_2} \cdots \sum_{j_{2N+1}=1}^{L_2} \\
& \times \exp\left\{ -\mathrm{i} \frac{t}{N+1} \sum_{n=1}^{N+1} H(p_{j_{2n}}, r_{i_{2n}}, t_n) \right\} \Bigg)_{i_{11},\ldots,i_{1N}, j_{11},\ldots,j_{1N+1}}, \qquad (13)
\end{aligned}
$$

where we have restricted the limits of the integration in configuration space between ρ_1 and ρ_2, and those of the integration in the momentum space between q_1 and q_2 but in fact they can be infinite. The j sums correspond to the momentum variables while the i sums correspond to the position variables. Moreover the group of i and j sums with index 2 corresponds to a certain small volume of phase space and the group of i and j sums with index 1 gather those small volumes. $M_1 + 1$ is the number of the partition points in each configuration space axis for which we get those small volumes and $M_2 + 1$ is the number of the partition points in each configuration space axis in those small volumes. $L_1 + 1$ is the number of the partition points in each momentum space axis for which we get those small volumes and $L_2 + 1$ is the number of the partition points in each momentum space axis in those small volumes. Therefore, if we apply the mean value theorem to those phase-space small volumes and let them shrink to points we recover the initial path integral.

To proceed we use Portmanteau's theorem [9] on equation (11) and therefore switch to integral relations on the small volumes of phase space with index 2 in equation (13). Then we gather those small volumes and transform back from equation (13) to equation (2) by shrinking the small volumes to points to obtain the following form:

$$
\begin{aligned}
K^{(N)}(r_f, r_i; t) = {} & \prod_{n=1}^{N} \left[\int \mathrm{d}r_n \right] \prod_{n=1}^{N+1} \left[\int \frac{\mathrm{d}p_n}{2\pi} \right] \exp\left\{ \mathrm{i} \sum_{n=1}^{N+1} [p_n(r_n - r_{n-1})] \right\} \exp(-\mathrm{i}t\,\langle H(t)\rangle) \\
& + \frac{1}{\sqrt{N+1}} \prod_{n=1}^{N} \left[\int \mathrm{d}r_n \right] \prod_{n=1}^{N+1} \left[\int \frac{\mathrm{d}p_n}{2\pi} \right] \exp\left\{ \mathrm{i} \sum_{n=1}^{N+1} [p_n(r_n - r_{n-1})] \right\} \\
& \times \left\{ \prod_{n=1}^{N+1} [f_1(p_n, r_n)] - \mathrm{i} \prod_{n=1}^{N+1} [g_1(p_n, r_n)] \right\}, \qquad (14a)
\end{aligned}
$$

where since the covariance matrices (12b) and (12c) are diagonal as we can easily check, we have set the Gaussian distributions of the random variables Y and W, as products of the functions

$$
\begin{aligned}
f_1(p_n, r_n) = {} & \frac{1}{2\pi \sigma_m \sigma_V(t) t^2 \sin^2(t\,\langle H(t)\rangle)} \\
& \times \exp\left\{ -\frac{p_n^2}{2\sigma_m^2 t^2 \sin^2(t\,\langle H(t)\rangle)} - \frac{r_n^2}{2\sigma_V^2(t) t^2 \sin^2(t\,\langle H(t)\rangle)} \right\} \qquad (14b)
\end{aligned}
$$

and

$$g_1(p_n, r_n) = \frac{1}{2\pi\sigma_m\sigma_V(t)t^2\cos^2(t\langle H(t)\rangle)} \times \exp\left\{-\frac{p_n^2}{2\sigma_m^2t^2\cos^2(t\langle H(t)\rangle)} - \frac{r_n^2}{2\sigma_V^2(t)t^2\cos^2(t\langle H(t)\rangle)}\right\} \tag{14c}$$

respectively. The mean values and the variances are calculated with respect to appropriate sampling functions for the momentum and the position relevant with the specific system studied.

Then on performing the integrations over all the p_n variables and the r_n variables on the first term in equation (14*a*) we obtain the theorem of the solution of the sign problem for equation (1), according to which the following expressions are equal as $N \to \infty$:

$$\begin{aligned} K^{(N)}(r_f, r_i; t) &= \prod_{n=1}^{N}\left[\int \mathrm{d}r_n\right]\prod_{n=1}^{N+1}\left[\int\frac{\mathrm{d}p_n}{2\pi}\right]\exp\left\{\mathrm{i}\sum_{n=1}^{N+1}[p_n(r_n - r_{n-1}) - \varepsilon H(p_n, r_n, t_n)]\right\} \\ &\cong \delta(r_f - r_i)\exp(-\mathrm{i}t\langle H(t)\rangle) + \frac{1}{\sqrt{N+1}}\prod_{n=1}^{N}\left[\int \mathrm{d}r_n\right] \\ &\quad\times\left\{\prod_{n=1}^{N+1}[f(r_n, r_{n-1})] - \mathrm{i}\prod_{n=1}^{N+1}[g(r_n, r_{n-1})]\right\}, \end{aligned} \tag{15a}$$

where

$$f(r_n, r_{n-1}) = \frac{1}{2\pi\sqrt{2\pi}\sigma_V(t)t\sin(t\langle H(t)\rangle)} \times \exp\left\{-\frac{1}{2}\sigma_m^2t^2\sin^2(t\langle H(t)\rangle)(r_n - r_{n-1})^2 - \frac{r_n^2}{2\sigma_V^2(t)t^2\sin^2(t\langle H(t)\rangle)}\right\} \tag{15b}$$

and

$$g(r_n, r_{n-1}) = \frac{1}{2\pi\sqrt{2\pi}\sigma_V(t)t\cos(t\langle H(t)\rangle)} \times \exp\left\{-\frac{1}{2}\sigma_m^2t^2\cos^2(t\langle H(t)\rangle)(r_n - r_{n-1})^2 - \frac{r_n^2}{2\sigma_V^2(t)t^2\cos^2(t\langle H(t)\rangle)}\right\}. \tag{15c}$$

Now we consider interacting systems. Therefore we can consider the double phase-space path integral

$$\begin{aligned} K_c(x_f, x_i; r_f, r_i; t) &= \iiiint Dr\frac{Dp}{2\pi}Dx\frac{Dp_x}{2\pi} \\ &\quad\times\exp\left[\mathrm{i}\int_0^t \mathrm{d}\tau\begin{pmatrix} p(\tau)\dot r(\tau) + p_x(\tau)\dot x(\tau) - H(p, r, \tau) \\ -H_x(p_x, x, \tau) - H_I(r, x, \tau)\end{pmatrix}\right] \\ &= \iint Dr\frac{Dp}{2\pi}h^{x_f, x_i, t}(r(\tau))\exp\left[\mathrm{i}\int_0^t \mathrm{d}\tau\,(p(\tau)\dot r(\tau) - H(p, r, \tau))\right], \end{aligned} \tag{16a}$$

where we have assumed ordinary path integrability over the x, p_x variables and we have set

$$h^{x_f, x_i, t}(r(\tau)) = \iint Dx\frac{Dp_x}{2\pi}\exp\left[\mathrm{i}\int_0^t \mathrm{d}\tau\,(p_x(\tau)\dot x(\tau) - H_x(p_x, x, \tau) - H_I(r, x, \tau))\right]. \tag{16b}$$

To proceed we can write the following expression which is similar to equation (13) and consider the same definitions as those appearing below equation (13):

$$K_c^{(N)}(r_f, r_i; t) = \frac{(\rho_2-\rho_1)^N}{M_1^N M_2^N} \frac{(q_2-q_1)^{N+1}}{(2\pi L_1 L_2)^{N+1}}$$
$$\sum_{i_{11}=1}^{M_1} \cdots \sum_{i_{1n}=1}^{M_1} \cdots \sum_{i_{1N}=1}^{M_1} \sum_{j_{11}=1}^{L_1} \cdots \sum_{j_{1n}=1}^{L_1} \cdots \sum_{j_{1N+1}=1}^{L_1}$$
$$\times \exp\left\{ \mathrm{i} \sum_{n=1}^{N+1} [p_{j_{1n}} (r_{i_{1n}} - r_{i_{1n-1}})] \right\} h(r_i, r_{i_{11}}, r_{i_{12}}, \ldots, r_{i_{1N}}, r_f)$$
$$\times \left(\sum_{i_{21}=1}^{M_2} \cdots \sum_{i_{2n}=1}^{M_2} \cdots \sum_{i_{2N}=1}^{M_2} \sum_{j_{21}=1}^{L_2} \cdots \sum_{j_{2n}=1}^{L_2} \cdots \sum_{j_{2N+1}=1}^{L_2} \right.$$
$$\left. \times \exp\left\{ -\mathrm{i} \frac{t}{N+1} \sum_{n=1}^{N+1} H(p_{j_{2n}}, r_{i_{2n}}, t_n) \right\} \right)_{i_{11},\ldots,i_{1N},j_{11},\ldots,j_{1N+1}}. \tag{17}$$

The only difference with equation (13) is the presence of the influence functional h. We follow the same discussion as in equations (5)–(14) and finally we use Portmanteau's theorem [9] on equation (11). Therefore we switch to integral relations on the small volumes of phase space with index 2 in equation (17). Then we transform back from equation (17) to the discrete form of equation (16*a*). So after the same manipulations which led from equation (14*a*) to equation (15*a*), we obtain the following final theorem of the solution of the sign problem for equation (16*a*), according to which the following expressions are equal as $N \to \infty$:

$$K_c^{(N)}(r_f, r_i; t) = \prod_{n=1}^{N} \left[\int \mathrm{d}r_n \right] \prod_{n=1}^{N+1} \left[\int \frac{\mathrm{d}p_n}{2\pi} \right] h(r_i, r_1, r_2, \ldots, r_N, r_f)$$
$$\times \exp\left\{ \mathrm{i} \sum_{n=1}^{N+1} [p_n(r_n - r_{n-1}) - \varepsilon H(p_n, r_n, t_n)] \right\}$$
$$\cong h(r_f, r_f, r_f, \ldots, r_f, r_f) \delta(r_f - r_i) \exp(-\mathrm{i}t \langle H(t) \rangle)$$
$$+ \frac{1}{\sqrt{N+1}} \prod_{n=1}^{N} \left[\int \mathrm{d}r_n \right] h(r_i, r_1, r_2, \ldots, r_N, r_f)$$
$$\times \left\{ \prod_{n=1}^{N+1} [f(r_n, r_{n-1})] - i \prod_{n=1}^{N+1} [g(r_n, r_{n-1})] \right\}. \tag{18}$$

Now we intend to prove that under appropriate conditions (see equation (27)) only the first term involving the delta function in equation (18) can give the exact result as $N \to \infty$. In that case we call that term *sign solved propagator* (SSP). To prove that, we intend to diagonalize and integrate the Gaussian products $\prod_{n=1}^{N+1} [f(r_n, r_{n-1})]$ and $\prod_{n=1}^{N+1} [g(r_n, r_{n-1})]$ under certain conditions for the h function in equation (18). To proceed we suppose that we intend to find the transition amplitude between the initial state $\Theta_i(r)$ and the final one $\Theta_f(r)$. Moreover to manage to find the eigenvalues of the quadratic forms in the Gaussian products and relate them to the roots of the Chebyshev polynomials of the second kind, we perform on the terms composed of the f functions the change of variables

$$r_n \to \frac{r_n}{\gamma_s(t)} \tag{19a}$$

and similarly we perform the change of variables

$$r_n \to \frac{r_n}{\gamma_c(t)} \tag{19b}$$

on the terms composed of the g functions.

We have set

$$\gamma_{\left\{ {s \atop c} \right\}}(t) = \frac{\sigma_m t \left\{ {\sin(t\langle H(t)\rangle) \atop \cos(t\langle H(t)\rangle)} \right\}}{\sqrt{2}}. \tag{19c}$$

Then upon taking into account the integration between the initial state $\Theta_i(r_i)$ and the final one $\Theta_f(r_f)$, the last term on the right-hand side of equation (18), involving the f and g functions, takes the form

$$\begin{aligned}
\wp = {} & \frac{1}{\sqrt{N+1}} \prod_{n=0}^{N+1} \left[\int \mathrm{d}r_n \right] \Theta_i\left(\frac{r_i}{\gamma_s(t)} \right) \Theta_f^*\left(\frac{r_f}{\gamma_s(t)} \right) \\
& \times h\left(\frac{r_i}{\gamma_s(t)}, \frac{r_1}{\gamma_s(t)}, \frac{r_2}{\gamma_s(t)}, \ldots, \frac{r_N}{\gamma_s(t)}, \frac{r_f}{\gamma_s(t)} \right) \\
& \times \frac{2^{\frac{N+2}{2}}}{[2\pi]^{N+1}[\sqrt{2\pi}\sigma_V(t)]^{N+1}\sigma_m^{N+2}[t\sin(t\langle H(t)\rangle)]^{2N+3}} \\
& \times \exp\left\{ r_i^2 + r_f^2 + \vec{\rho}_1 \overleftrightarrow{M}_{N+2}(\beta_s(t))\vec{\rho}_1^{\,T} \right\} \\
& - \mathrm{i}\frac{1}{\sqrt{N+1}} \prod_{n=0}^{N+1} \left[\int \mathrm{d}r_n \right] \Theta_i\left(\frac{r_i}{\gamma_c(t)} \right) \Theta_f^*\left(\frac{r_f}{\gamma_c(t)} \right) \\
& \times h\left(\frac{r_i}{\gamma_c(t)}, \frac{r_1}{\gamma_c(t)}, \frac{r_2}{\gamma_c(t)}, \ldots, \frac{r_N}{\gamma_c(t)}, \frac{r_f}{\gamma_c(t)} \right) \\
& \times \frac{2^{\frac{N+2}{2}}}{[2\pi]^{N+1}[\sqrt{2\pi}\sigma_V(t)]^{N+1}\sigma_m^{N+2}[t\cos(t\langle H(t)\rangle)]^{2N+3}} \\
& \times \exp\left\{ r_i^2 + r_f^2 + \vec{\rho}_1 \overleftrightarrow{M}_{N+2}(\beta_c(t))\vec{\rho}_1^{\,T} \right\},
\end{aligned} \tag{20a}$$

where

$$\vec{\rho}_1 = (r_i, r_1, \ldots r_N, r_f), \tag{20b}$$

$$\beta_{\left\{ {s \atop c} \right\}}(t) = -2 - \frac{1}{\sigma_m^2\sigma_V^2(t)t^4 \left\{ {\sin^4(t\langle H(t)\rangle) \atop \cos^4(t\langle H(t)\rangle)} \right\}}. \tag{20c}$$

Moreover the matrices on the exponents in equation (20a) correspond to the symmetric matrices

$$\overleftrightarrow{M}_0(\beta) = 1 \tag{21a}$$

$$\overleftrightarrow{M}_1(\beta) = [\beta] \tag{21b}$$

$$\overleftrightarrow{M}_2(\beta) = \begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix} \tag{21c}$$

and generally

$$(\overleftrightarrow{M}_{N+2}(\beta))_{ij} = \begin{cases} 1 & \text{if } i = j \pm 1 \\ \beta & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{21d}$$

Their determinates satisfy the recurrence relation

$$\det(\overset{\leftrightarrow}{M}_{N+1}(\beta)) = \beta \det(\overset{\leftrightarrow}{M}_N(\beta)) - \det(\overset{\leftrightarrow}{M}_{N-1}(\beta)), \tag{22a}$$

and therefore [10] if $U_{N+2}(x)$ is a Chebyshev polynomial of the second kind of order $N + 2$, then

$$\det(\overset{\leftrightarrow}{M}_{N+2}(\beta)) = U_{N+2}\left(\frac{\beta}{2}\right). \tag{22b}$$

So we easily conclude that the eigenvalues λ of the matrix $\overset{\leftrightarrow}{M}_{N+2}(\beta)$ of order $N + 2$ can be calculated from the $N + 2$ solutions of the equation

$$U_{N+2}\left(\frac{\beta-\lambda}{2}\right) = 0 \tag{23}$$

More particularly let the numbers $\xi^{*(n)}_{N+2}$, $n = 0, \ldots, N + 1$, be the roots of the equation $U_{N+2}(x) = 0$. They are simple, real roots and $\xi^{*(n)}_{N+2} \in (-1, 1)$, $n = 0, \ldots, N + 1$. Then the eigenvalues of the matrices $\overset{\leftrightarrow}{M}_{N+2}\,(\beta_c(t))$ and $\overset{\leftrightarrow}{M}_{N+2}\,(\beta_s(t))$ are going to be given respectively by the expressions

$$\lambda^{*(N+2)}_{\left\{\substack{s\\c}\right\}n} = -2\xi^{*(n)}_{N+2} + \beta_{\left\{\substack{s\\c}\right\}}(t)\,. \tag{24}$$

Moreover the diagonal quadratic form corresponding to the term $r_i^2 + r_f^2$ on the exponents in equation (20a) can be diagonalized simultaneously with each one of the quadratic forms corresponding to the matrices $\overset{\leftrightarrow}{M}_{N+2}\,(\beta_c(t))$ and $\overset{\leftrightarrow}{M}_{N+2}\,(\beta_s(t))$ and we finally conclude that the eigenvalues of the full quadratic forms on the exponents in equation (20a) are going to have the form

$$\lambda^{(N+2)}_{\left\{\substack{s\\c}\right\}n} = -2\xi^{(n)}_{\left\{\substack{s\\c}\right\}N+2} + \beta_{\left\{\substack{s\\c}\right\}}(t), \tag{25}$$

where

$$\xi^{(n)}_{\left\{\substack{s\\c}\right\}N+2} = \xi^{*(n)}_{N+2} + \sigma^{(n)}_{\left\{\substack{s\\c}\right\}N+2} \tag{26}$$

and $\sigma^{(n)}_{cN+2}$, $\sigma^{(n)}_{sN+2}$ are appropriate non-negative real numbers.

To proceed further we have to make certain assumptions on the function h in equation (20a). For instance, in the application of [6] the h function is composed of a product of a certain bounded combination of error functions. Therefore it is reasonable to assume the condition

$$\left|h\left(\frac{r_i}{\gamma(t)}, \frac{r_1}{\gamma(t)}, \frac{r_2}{\gamma(t)}, \ldots, \frac{r_N}{\gamma(t)}, \frac{r_f}{\gamma(t)}\right)\right| \leqslant b(N+1)^{\gamma_1} C^{N+2} \qquad b, C > 0, \quad \gamma_1 \in R, \tag{27}$$

where $\gamma(t) = \gamma_c(t)$ or $\gamma(t) = \gamma_s(t)$.

Then if the range of the r_n variables is from $-\infty$ to ∞ we obtain the following upper bound of the expression $\wp$ in equation (20a):

$$\begin{aligned} |\wp| \leqslant{}& bb_1 2\pi\sqrt{2\pi}\sigma_V(t)\,|t \sin(t\,\langle H(t)\rangle)| \left\{\frac{(N+1)^{\gamma_1}}{\sqrt{N+1}} \prod_{n=0}^{N+1}\left[\frac{C}{2\pi\sqrt{\Lambda^{(N+2)}_{sn}}}\right]\right\} \\ &+ bb_2 2\pi\sqrt{2\pi}\sigma_V(t)|t\cos(t\langle H(t)\rangle)| \left\{\frac{(N+1)^{\gamma_1}}{\sqrt{N+1}} \prod_{n=0}^{N+1}\left[\frac{C}{2\pi\sqrt{\Lambda^{(N+2)}_{cn}}}\right]\right\}, \end{aligned} \tag{28a}$$

where

$$\Lambda^{(N+2)}_{\{{s \atop c}\}n} = 1 + 2\left(1 + \xi^{(n)}_{\{{s \atop c}\}N+2}\right)\sigma_m^2\sigma_V^2(t)t^4 \begin{Bmatrix} \sin^4(t\langle H(t)\rangle) \\ \cos^4(t\langle H(t)\rangle) \end{Bmatrix}. \tag{28b}$$

The constants b_1, b_2 depend on the form of the initial and final wavefunctions $\Theta_i(r)$ and $\Theta_f(r)$.

Therefore if the expressions in the curly brackets in equation (28a) tend to zero as $N \to \infty$, then the first term in equation (18) is exact as $N \to \infty$ and corresponds to the sign solved propagator. In conclusion we have proven the sign solved propagator theorem according to which if equations (16a) and (16b) as well as equation (27) are valid and moreover

$$\lim_{N\to\infty}\left\{\frac{(N+1)^{\gamma_1}}{\sqrt{N+1}}\prod_{n=0}^{N+1}\left[\frac{C}{2\pi\sqrt{\Lambda^{(N+2)}_{sn}}}\right]\right\} = \lim_{N\to\infty}\left\{\frac{(N+1)^{\gamma_1}}{\sqrt{N+1}}\prod_{n=0}^{N+1}\left[\frac{C}{2\pi\sqrt{\Lambda^{(N+2)}_{cn}}}\right]\right\} = 0, \tag{29a}$$

then

$$K_c(x_f, x_i; r_f, r_i; t) = \delta(r_f - r_i)\exp(-\mathrm{i}t\langle H(t)\rangle)\lim_{N\to\infty} h^{x_f,x_i,t}(\underbrace{r_f, r_f, r_f, \ldots, r_f, r_f}_{N+2}). \tag{29b}$$

Expressions (29a) are valid if $C < 2\pi$, since $\Lambda^{(N+2)}_{\{{s \atop c}\}n} \geqslant 1$ (see equation (28b)).

3. Application to two coupled and forced harmonic oscillators

Let us consider the following Hamiltonian of two coupled and forced harmonic oscillators of time-dependent frequencies and masses and coupling of the form $x_1x_2^\alpha$,

$$H(t) = \sum_{j=1}^{2}\left[\frac{p_j^2}{2m_j(t)} + \frac{1}{2}m_j(t)\omega_j^2(t)x_j^2 - m_j(t)f_j(t)x_j\right] - \lambda(t)x_1x_2^\alpha. \tag{30}$$

We intend to evaluate its sign solved propagator (SSP).

As a first step we perform the linear transformation

$$x_i = \frac{Q_i}{\sqrt{m_i(t)}}, \qquad i = 1, 2 \tag{31a}$$

$$p_i = \sqrt{m_i(t)}\left[P_i - \frac{\dot{m}_i(t)}{2\sqrt{m_i(t)}}x_i\right], \qquad i = 1, 2, \tag{31b}$$

which is canonical as it preserves the Poisson brackets [11]. Then we obtain the free of mass terms Hamiltonian

$$H_0(t) = \sum_{j=1}^{2}\left[\frac{P_j^2}{2} + \frac{1}{2}\Omega_j^2(t)Q_j^2 - F_j(t)Q_j\right] - \Gamma(t)Q_1Q_2^\alpha, \tag{32a}$$

where

$$\Omega_i^2(t) = \left[\omega_i^2(t) + \frac{1}{4}\left(\frac{\dot{m}_i^2(t)}{m_i^2(t)} - 2\frac{\ddot{m}_i(t)}{m_i(t)}\right)\right] \qquad i = 1, 2 \tag{32b}$$

$$F_i(t) = \sqrt{m_i(t)}f_i(t) \qquad i = 1, 2 \tag{32c}$$

$$\Gamma(t) = \frac{\lambda(t)}{\sqrt{m_1(t)m_2^\alpha(t)}}. \tag{32d}$$

If $K(x_{1f}, x_{2f}, t_f; x_{1i}, x_{2i}, t_i)$ is the propagator corresponding to the Hamiltonian (30) and $K_0(Q_{1f}, Q_{2f}, t_f; Q_{1i}, Q_{2i}, t_i)$ is the one corresponding to the Hamiltonian (32*a*) then they are related up to a surface factor [11] since the Hamiltonian (32*a*) has appeared from the Hamiltonian (30) under a canonical transformation. Moreover canonical transformations preserve the volume element [11] in phase space. Therefore we easily find the relation

$$K(x_{1f}, x_{2f}, t_f; x_{1i}, x_{2i}, t_i) = \prod_{j=1}^{2} [m_j(t_f) m_j(t_i)]^{\frac{1}{4}} \exp\left[-\frac{\mathrm{i}}{4}\left(\dot{m}_j(t_f) x_{jf}^2 - \dot{m}_j(t_i) x_{ji}^2\right)\right] \times K_0(Q_{1f}, Q_{2f}, t_f; Q_{1i}, Q_{2i}, t_i). \tag{33}$$

Therefore we can concentrate our attention on the propagator of the Hamiltonian (32*a*). The SSP can be evaluated via ordinary path integration over the variables 1 and then by application of the theory of section 2. To proceed towards the first integration we intend to perform a canonical transformation, followed by a time transformation, defined as

$$Q_1 = X\rho(t) \tag{34a}$$

$$P_1 = \frac{P}{\rho(t)} \tag{34b}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \rho^{-2}(t). \tag{34c}$$

It is canonical since it preserves the Poisson brackets and therefore it preserves the volume element in phase space [11]. As the present transformation involves the generic time redefinition (34*c*) we give more details. The $N + 1$ time slices discrete form of $K_0(Q_{1f}, Q_{2f}, t_f; Q_{1i}, Q_{2i}, t_i)$ involves the times $t_n = t_i + n\varepsilon$, $n = 0, 1, \ldots, N + 1$, where the time step is $\varepsilon = \frac{t_f - t_i}{N+1}$. Now on integrating the path integral expression on the momentums, it becomes (see below as well)

$$K_0(Q_{1f}, Q_{2f}, t_f; Q_{1i}, Q_{2i}, t_i) = \left(\frac{1}{2\pi \mathrm{i}\varepsilon}\right)^{N+1} \int \prod_{n=1}^{N} [\mathrm{d}Q_{1n}\, \mathrm{d}Q_{2n}] \exp[\mathrm{i}S^{(N)}]. \tag{35a}$$

Then under the transformations (34*a*)–(34*c*) the time step becomes $\sigma_n = \frac{\varepsilon}{\rho(t_n)\rho(t_{n-1})}$, where we have symmetrized the expression in order to avoid any preference of the one time over the other. So we conclude that the path differential measure takes the form

$$\begin{aligned}
&\left(\frac{1}{2\pi \mathrm{i}\varepsilon}\right)^{N+1} \prod_{n=1}^{N} [\mathrm{d}Q_{1n}\, \mathrm{d}Q_{2n}] \\
&\quad = \left(\frac{1}{2\pi \mathrm{i}\varepsilon}\right)^{\frac{(N+1)}{2}} \prod_{n=1}^{N+1} \left(\frac{1}{2\pi \mathrm{i}\sigma_n \rho(t_n)\rho(t_{n-1})}\right)^{\frac{1}{2}} \prod_{n=1}^{N} [\rho_n(t_n)\, \mathrm{d}X_{1n}\, \mathrm{d}Q_{2n}] \\
&\quad = \frac{1}{(\rho_f \rho_i)^{\frac{1}{2}}} \left(\frac{1}{2\pi \mathrm{i}\varepsilon}\right)^{\frac{(N+1)}{2}} \prod_{n=1}^{N+1} \left(\frac{1}{2\pi \mathrm{i}\sigma_n}\right)^{\frac{1}{2}} \prod_{n=1}^{N} [\mathrm{d}X_{1n}\, \mathrm{d}Q_{2n}],
\end{aligned} \tag{35b}$$

and the discretized action appearing in equation (35*a*) is

$$S^{(N)} = \sum_{n=1}^{N+1} \left[\begin{array}{l} \frac{(Q_{1n} - Q_{1n-1})^2}{2\varepsilon} + \frac{(Q_{2n} - Q_{2n-1})^2}{2\varepsilon} - \varepsilon\left(\frac{1}{2}\Omega_2^2(t_n) Q_{2n}^2 - F_2(t_n) Q_{2n}\right) \\ -\varepsilon\left(\frac{1}{2}\Omega_1^2(t_n) Q_{1n}^2 - F_1(t_n) Q_{1n}\right) + \varepsilon\Gamma(t_n) Q_{1n} Q_{2n}^{\alpha} \end{array}\right]$$

$$= \sum_{n=1}^{N+1} \left[\begin{array}{l} \frac{(Q_{2n}-Q_{2n-1})^2}{2\sigma_n \rho(t_n)\rho(t_{n-1})} - \sigma_n \rho(t_n)\rho(t_{n-1}) \left(\frac{1}{2}\Omega_2^2(t_n) Q_{2n}^2 - F_2(t_n) Q_{2n}\right) \\ + \frac{\bar\rho(s_n) X_n^2}{2\sigma_n \bar\rho(s_{n-1})} + \frac{\bar\rho(s_{n-1}) X_{n-1}^2}{2\sigma_n \bar\rho(s_n)} - \frac{X_n X_{n-1}}{\sigma_n} - \sigma_n \begin{pmatrix} \frac{1}{2}\Omega_1^2(s_n)\bar\rho(s_{n-1})\bar\rho^3(s_n) X_n^2 \\ -F_1(s_n)\bar\rho(s_{n-1})\bar\rho^2(s_n) X_n \end{pmatrix} \\ + \sigma_n \Gamma(s_n)\bar\rho(s_{n-1})\bar\rho^2(s_n) X_n Q_{2n}^{\alpha} \end{array} \right]. \tag{35c}$$

Therefore on using the expansions $\frac{\bar\rho(s)}{\bar\rho(s\pm\sigma)} = 1 \mp \frac{\dot{\bar\rho}(s)}{\bar\rho(s)}\sigma + \left(\frac{\dot{\bar\rho}^2(s)}{\bar\rho^2(s)} - \frac{\ddot{\bar\rho}(s)}{2\bar\rho(s)}\right)\sigma^2 + O(\sigma^3)$ we find that the propagator $K_0(Q_{1f}, Q_{2f}, t_f; Q_{1i}, Q_{2i}, t_i)$ is related to the transformed one as

$$K_0(Q_{1f}, Q_{2f}, t_f; Q_{1i}, Q_{2i}, t_i) = \frac{1}{(\rho_f \rho_i)^{\frac{1}{2}}} \exp\left\{ \frac{\mathrm{i}}{2} \left(\frac{\dot{\bar\rho}_f}{\rho_f} X_f^2 - \frac{\dot{\bar\rho}_i}{\rho_i} X_i^2 \right) \right\} K_0(X_{1f}, Q_{2f}, s_f; X_{1i}, Q_{2i}, s_i), \tag{35d}$$

where on switching to a phase-space path integral we have

$$\begin{aligned} K_0(X_{1f}, Q_{2f}, s_f; X_{1i}, Q_{2i}, s_i) &= \iint D Q_2 \frac{D P_2}{2\pi} D X \frac{D \mathrm{P}}{2\pi} \\ &\times \exp\left\{ \mathrm{i} \int_{s_i}^{s_f} ds \left[\rho^2(t) \left[P_2 \dot{Q}_2 - \left(\frac{P_2^2}{2} + \frac{1}{2}\Omega_2^2(t) Q_2^2 - F_2(t) Q_2 \right) \right] \right.\right. \\ &\left.\left. + P\dot{X} - \left(\frac{P^2}{2} + \frac{1}{2}\left(\tilde\Omega^2 + \bar\Omega_1^2(s)\bar\rho^4\right) X^2 - \bar F(s)\bar\rho^3 X \right) \right] \right\}. \end{aligned} \tag{36}$$

Moreover $t = t(s)$,

$$\tilde\Omega^2 = \left[\frac{\ddot{\bar\rho}}{\bar\rho} - 2\left(\frac{\dot{\bar\rho}}{\bar\rho}\right)^2 \right] = \rho^3 \ddot\rho, \tag{37}$$

and we have used the notations

$$\rho = \bar\rho(s) = \rho(t) \tag{38a}$$

$$\dot\rho = \frac{\mathrm{d}\rho}{\mathrm{d}t} \tag{38b}$$

$$\dot{\bar\rho} = \frac{\mathrm{d}\bar\rho}{\mathrm{d}s} \tag{38c}$$

$$\Omega_1^2(t) = \bar\Omega_1^2(s) \tag{38d}$$

$$\bar F(s) = F(t) = F_1(t) + \Gamma(t) Q_2^{\alpha}. \tag{38e}$$

Now we impose a constrain on ρ by setting the global time-dependent frequency multiplying X^2 in equation (36) equal to a constant

$$\tilde\Omega^2 + \bar\Omega_1^2(s)\bar\rho^4 = \omega_0^2 = \text{const}. \tag{39}$$

Therefore the integration with respect to the (X, P) variables corresponds to a forced harmonic oscillator with constant frequency. We find the propagator

$$\begin{aligned} K_0(Q_{1f}, Q_{2f}, t_f; Q_{1i}, Q_{2i}, t_i) &= \int D Q_2 \frac{D P_2}{2\pi} h(Q_2(t), t; Q_{1f}, t_f; Q_{1i}, t_i) \\ &\times \exp\left\{ \mathrm{i} \int_{t_i}^{t_f} \mathrm{d}t \left[P_2 \dot{Q}_2 - \left(\frac{P_2^2}{2} + \frac{1}{2}\Omega_2^2(t) Q_2^2 - F_2(t) Q_2 \right) \right] \right\}, \end{aligned} \tag{40}$$

where

$$h(Q_2(t), t; Q_{1f}, t_f; Q_{1i}, t_i) = \sqrt{\frac{\omega_0}{2\pi \mathrm{i} \rho_f \rho_i \sin\varphi(t_f, t_i)}} \exp\left\{\frac{\mathrm{i}}{2}\left(\frac{\dot\rho_f}{\rho_f} Q_{1f}^2 - \frac{\dot\rho_i}{\rho_i} Q_{1i}^2\right)\right\}$$
$$\times \exp\left\{\frac{\mathrm{i}\omega_0}{2\sin\varphi\left(t_f, t_i\right)}\left[\begin{array}{l}\left(\frac{Q_{1f}^2}{\rho_f^2} + \frac{Q_{1i}^2}{\rho_i^2}\right)\cos\varphi(t_f, t_i) - \frac{2Q_{1f}Q_{1i}}{\rho_f\rho_i}\\ +\frac{2}{\omega_0}\frac{Q_{1f}}{\rho_f}\int_{t_i}^{t_f} G(t)\sin\varphi(t, t_i)\,\mathrm{d}t + \frac{2}{\omega_0}\frac{Q_{1i}}{\rho_i}\int_{t_i}^{t_f} G(t)\sin\varphi(t_f, t)\,\mathrm{d}t\\ -\frac{2}{\omega_0^2}\int_{t_i}^{t_f}\mathrm{d}t\int_{t_i}^{t}\mathrm{d}\tau\, G(t)G(\tau)\sin\varphi(t_f, t)\sin\varphi(\tau, t_i)\end{array}\right]\right\} \tag{41}$$

$$G(t) = \left(F_1(t) + \Gamma(t)Q_2^\alpha\right)\rho(t) \tag{42}$$

$$\varphi(t'', t') = \omega_0 \int_{t'}^{t''} \frac{\mathrm{d}t}{\rho^2(t)}, \tag{43}$$

and $\rho(t)$ is the solution of the differential equation

$$\ddot\rho + \Omega_1^2(t)\rho = \frac{\omega_0^2}{\rho^3}. \tag{44}$$

Now we are in the position to apply the theory of section 2. At first we consider $N+1$ time slices and further appropriate sampling functions for the position and the momentum which in fact can be the squared measure of a corresponding wavefunction $\Psi_2(Q_2) = \langle Q_2 \mid \Psi_2\rangle$ for the configuration space and of its Fourier transform function $\Psi_2(P_2) = \langle P_2 \mid \Psi_2\rangle$ for the momentum space (see the discussion below equation (4)). That wavefunction for instance can be the ground state of a relevant harmonic oscillator and it does not appear in most final results as it appears in just a phase term (see the discussion in section 4). Then if the $G(t)$ function given by equation (42) is real, as expected, the h function in equation (40) (compare with equations (16*a*) and (16*b*)), which is given by equation (41), is bounded by unity up to a constant (i.e. $C = 1$ and $\gamma_1 = 0$ in the corresponding equation (27)) and therefore we are able to apply expression (29*b*) and obtain

$$K_0(Q_{1f}, Q_{2f}, t_f; Q_{1i}, Q_{2i}, t_i) = \exp\left\{-\mathrm{i}\int_{t_i}^{t_f}\mathrm{d}t\langle\Psi_2|\frac{P_2^2}{2} + \frac{1}{2}\Omega_2^2(t)Q_2^2 - F_2(t)Q_2|\Psi_2\rangle\right\}$$
$$\times\delta(Q_{2f} - Q_{2i})\sqrt{\frac{\omega_0}{2\pi\mathrm{i}\rho_f\rho_i\sin\varphi(t_f, t_i)}}\exp\left\{\frac{\mathrm{i}}{2}\left(\frac{\dot\rho_f}{\rho_f}Q_{1f}^2 - \frac{\dot\rho_i}{\rho_i}Q_{1i}^2\right)\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}\omega_0}{2\sin\varphi(t_f, t_i)}\left[\begin{array}{l}\left(\frac{Q_{1f}^2}{\rho_f^2} + \frac{Q_{1i}^2}{\rho_i^2}\right)\cos\varphi(t_f, t_i) - \frac{2Q_{1f}Q_{1i}}{\rho_f\rho_i}\\ +\frac{2}{\omega_0}\frac{Q_{1f}}{\rho_f}\int_{t_i}^{t_f}\left(F_1(t) + \Gamma(t)Q_{2f}^\alpha\right)\rho(t)\sin\varphi(t, t_i)\,\mathrm{d}t\\ +\frac{2}{\omega_0}\frac{Q_{1i}}{\rho_i}\int_{t_i}^{t_f}\left(F_1(t) + \Gamma(t)Q_{2f}^\alpha\right)\rho(t)\sin\varphi(t_f, t)\,\mathrm{d}t\\ -\frac{2}{\omega_0^2}\int_{t_i}^{t_f}\mathrm{d}t\int_{t_i}^{t}\mathrm{d}\tau\left[\begin{array}{l}\left(F_1(t) + \Gamma(t)Q_{2f}^\alpha\right)\left(F_1(\tau) + \Gamma(\tau)Q_{2f}^\alpha\right)\\ \times\rho(t)\rho(\tau)\sin\varphi(t_f, t)\sin\varphi(\tau, t_i)\end{array}\right]\end{array}\right]\right\}. \tag{45}$$

We can obtain the final SSP $K(x_{1f}, x_{2f}, t_f; x_{1i}, x_{2i}, t_i)$ from the above expression by taking into account equations (31*a*), (32*b*)–(32*d*) and (33).

So we have found the SSP of the very general Hamiltonian (30) under the only assumption that we know a solution of equation (44) [12].

Now we proceed towards a more specific application of the general expression (45). So we assume the following specific form for the Hamiltonian (30):

$$
\begin{aligned}
H(t) = {} & \mathrm{e}^{-\gamma_1 t}\frac{p_1^2}{2m} + \mathrm{e}^{\gamma_1 t}\left(\frac{1}{2}m\omega_1^2 x_1^2 - x_1 m f_1(t)\right) \\
& + \mathrm{e}^{-\gamma_2 t}\frac{p_2^2}{2m} + \mathrm{e}^{\gamma_2 t}\left(\frac{1}{2}m\omega_2^2 x_2^2 - x_2 m f_2(t)\right) - \lambda(t) x_1 x_2,
\end{aligned} \tag{46}
$$

where we have set $\alpha = 1$. Then according to the above method we extract the following form for the SSP:

$$
\begin{aligned}
& K(x_{1f}, x_{2f}, t; x_{1i}, x_{2i}, 0) \\
& = \exp\left\{-\mathrm{i}\int_0^t \mathrm{d}\tau \langle\Psi_2|\, \mathrm{e}^{-\gamma_2\tau}\frac{p_2^2}{2m} + \mathrm{e}^{\gamma_2\tau}\left(\frac{1}{2}m\omega_2^2 x_2^2 - x_2 m f_2(\tau)\right)|\Psi_2\rangle\right\} \\
& \times \delta\left(\mathrm{e}^{\gamma_2 t/2}x_{2f} - x_{2i}\right)\sqrt{\frac{m\omega_1\,\mathrm{e}^{\gamma_1 t/2}}{2\pi i \sin\omega_1 t}} \\
& \times \exp\left\{\frac{im}{2}\left[\begin{array}{l}
\left(-\frac{1}{2}\gamma_1 + \omega_1\cot\omega_1 t\right)\mathrm{e}^{\gamma_1 t}x_{1f}^2 + \left(\frac{1}{2}\gamma_1 + \omega_1\cot\omega_1 t\right)x_{1i}^2 - \frac{2\omega_1}{\sin\omega_1 t}\mathrm{e}^{\gamma_1 t/2}x_{1i}x_{1f} \\
+\frac{2x_{1f}}{m\sin\omega_1 t}\mathrm{e}^{\gamma_1 t/2}\int_0^t \mathrm{d}\tau\left(mf_1(\tau) + \lambda(\tau)\,\mathrm{e}^{-\gamma_1\tau}x_{2f}\,\mathrm{e}^{\frac{\gamma_2(t-\tau)}{2}}\right)\mathrm{e}^{\frac{\gamma_1\tau}{2}}\sin(\omega_1\tau) + \\
\frac{2x_{1i}}{m\sin\omega_1 t}\int_0^t \mathrm{d}\tau\left(mf_1(\tau) + \lambda(\tau)\,\mathrm{e}^{-\gamma_1\tau}x_{2f}\,\mathrm{e}^{\frac{\gamma_2(t-\tau)}{2}}\right)\mathrm{e}^{\frac{\gamma_1\tau}{2}}\sin(\omega_1(t-\tau)) - \frac{1}{2m^2\omega_1} \\
\times\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}\rho\left[\begin{array}{l}\left(mf_1(\tau) + \lambda(\tau)\,\mathrm{e}^{-\gamma_1\tau}x_{2f}\,\mathrm{e}^{\frac{\gamma_2(t-\tau)}{2}}\right)\left(mf_1(\rho) + \lambda(\rho)\,\mathrm{e}^{-\gamma_1\rho}x_{2f}\,\mathrm{e}^{\frac{\gamma_2(t-\rho)}{2}}\right) \\ \times\mathrm{e}^{\gamma_1(\tau+\rho)/2}\sin(\omega_1(t-\tau))\sin(\omega_1\rho)\end{array}\right]
\end{array}\right]\right\}.
\end{aligned} \tag{47}
$$

The above result can be applied to any standard calculation that involves propagators.

4. Conclusions

In conclusion, in the present paper, we have given a method which uses the central limit theorem to manage the handling of path integral expressions. The extracted results correspond to a completely new formalism and cannot be directly related to previous results. The power of the present method is that it is capable of giving closed expressions even in the case of systems totally coupled and interacting with the requirements, on the one hand to be able to path integrate on the first of the two systems at least after an integral transform, and on the other, conditions like equations (27) and (29*a*) to be valid. The second system can be of any form and we just have to know one of its states relevant with the dynamical situation studied. We further note that that state appears in just a phase and so in most cases including the calculation of the geometric phase and the evaluation of phase-space representations, such as the Wigner, Husimi, Glauber, Kirkwood ones, that state does not appear anywhere in the final results and therefore we can obtain exact closed expressions.

As an application we have considered two coupled harmonic oscillators. We note that Hamiltonians of the form (30) may appear after appropriate integral transforms of more complex interactions in relevant path integrals expressions.

Concluding we think that the present paper's method is tractable and adds new knowledge in mathematical physics.

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